On regular irreducible components of module varieties over string algebras

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Abstract

We determine the regular irreducible components of the variety $\mathrm{mod}(\mathcal{A}, d)$, where $\mathcal{A} = kQ/I$ is a string algebra and I is generated by a set of paths of length two. Our case is among the first examples of descriptions of irreducible components, aside from hereditary, tubular (see [4]) and Gelfand-Ponomarev algebras (see [7]).

1 Introduction

Fix an algebraically closed field k, and let $\mathcal{A}$ be a finite dimensional associative k-algebra with a unit. By $\mathrm{mod}(\mathcal{A}, d)$ we denote the affine subvariety of $\mathrm{Hom}_k(\mathcal{A}, M_d(k))$ of k-algebra homomorphisms $A \longrightarrow M_d(k)$, where $M_d(k)$ is the algebra of $d \times d$ matrices over k. The algebraic group $\mathrm{GL}_d(k)$ of invertible $d \times d$ matrices acts on $\mathrm{mod}(\mathcal{A}, d)$ by conjugation. The $\mathrm{GL}_d(k)$-orbits in $\mathrm{mod}(\mathcal{A}, d)$ are in one-to-one correspondence with the isomorphism classes of d-dimensional left $\mathcal{A}$-modules.

We recall the definition of a string algebra and describe its finite dimensional modules. A quiver is a tuple $Q = (Q_0, Q_1, s, t)$ consisting of a finite set of vertices Q_0, a finite set of arrows Q_1 and maps $s, t : Q_1 \longrightarrow Q_0$. We call $s(\alpha)$ the source and $t(\alpha)$ the target of the arrow $\alpha \in Q_1$. A path in Q is either 1_u, where u is a vertex of Q, or a finite sequence $\alpha_1 \ldots \alpha_n$ of arrows of Q, satisfying $s(\alpha_i) = t(\alpha_{i+1})$. The path algebra kQ is the k-algebra with k-basis $\{p : p \text{ is a path in } Q\}$, where the product of two paths p, q is the concatenation pq in case $s(p) = t(q)$ and zero otherwise. A string algebra $\mathcal{A}$ over k is an algebra of the form $\mathcal{A} = kQ/I$ for a quiver Q and an ideal I in kQ satisfying the following:

- I is admissible, i.e. there is an $n \in \mathbb{N}$ with $(kQ^+)^n \subseteq I \subseteq (kQ^+)^2$, where kQ^+ is the ideal generated by all arrows.

- I is generated by a set of paths in Q.

- There are at most two arrows with source u for any vertex u of Q.

- There are at most two arrows with target u for any vertex u of Q.

- For any arrow α of Q there is a most one arrow β with $s(\alpha) = t(\beta)$ and $\alpha\beta \notin I$.

- For any arrow β of Q there is a most one arrow α with $s(\alpha) = t(\beta)$ and $\alpha\beta \notin I$.

Fix a string algebra $\mathcal{A} = kQ/I$. The opposite quiver Q^{op} is (Q_0, Q_1^{-1}, s, t), where $Q_1^{-1} = \{\alpha^{-1} : \alpha \in Q_1\}$, $s(\alpha^{-1}) := t(\alpha)$ and $t(\alpha^{-1}) := s(\alpha)$. A string (of $\mathcal{A}$) is either 1_u, where u is a vertex of Q, or a finite sequence $\alpha_1 \cdots \alpha_n$ of arrows of Q and Q^{op}, satisfying $s(\alpha_i) = t(\alpha_{i+1})$ and $\alpha_i \neq \alpha_{i+1}^{-1}$ such that none of its partial strings $\alpha_i \cdots \alpha_j$ nor its inverse $\alpha_j^{-1} \cdots \alpha_i^{-1}$ belongs to I. By $\mathcal{W}$ we denote the set of all strings. Let c be a string. We set $s(c) := t(c) := u$ in case $c = 1_u$ and $s(c) := s(\alpha_n)$ and $t(c) = t(\alpha_1)$, in case $c = \alpha_1 \cdots \alpha_n$. We say that c starts/ends with an (inverse) arrow if $c = \alpha_1 \cdots \alpha_n$ and $\alpha_n \in Q_1$ ($\in Q_1^{-1}$), $\alpha_1 \in Q_1$ ($\in Q_1^{-1}$), respectively. We define the inverse string c^{-1} of c by $c^{-1} := 1_u$ in case $c = 1_u$ and $c^{-1} := \alpha_n^{-1} \cdots \alpha_1^{-1}$ if $c = \alpha_1 \cdots \alpha_n$. The length $l(c)$ of c is 0 if $c = 1_u$ and n if $c = \alpha_1 \cdots \alpha_n$. We call the strings of length 0 trivial. Note that the concatenation cd of two strings c and d is not necessarily a string.

Remark 1.1.

- *A string c is trivial if and only if $c = c^{-1}$.*

- *If I is generated by paths of length two, then $c = \alpha_1 \cdots \alpha_n$ is a string if and only if $\alpha_i\alpha_{i+1}$ is a string for all $1 \leq i \leq n-1$.*

By $\mathrm{Ld}(c) := \{c' : c = c'c''\}$ we denote the set of leftdivisors of c. For any string c we define the string module $M(c)$ with basis
$\{e_{c'} : c' \in \mathrm{Ld}(c)\}$ by

$$p \cdot e_{c'} = \begin{cases} e_{c'p^{-1}} & \text{if } c'p^{-1} \in \mathrm{Ld}(c), \\ e_{c''} & \text{if } c' = c''p, \\ 0 & \text{otherwise,} \end{cases}$$

for any path p in Q. For an arrow α and a vertex u we thus have

$$e_{\alpha_1\cdots\alpha_{i-1}} \xrightarrow{\alpha} e_{\alpha_1\cdots\alpha_i} \qquad \text{if } \alpha = \alpha_i^{-1},$$

$$e_{\alpha_1\cdots\alpha_{i-1}} \xleftarrow{\alpha} e_{\alpha_1\cdots\alpha_i} \qquad \text{if } \alpha = \alpha_i,$$

$$1_u \cdot e_{\alpha_1\cdots\alpha_i} = \begin{cases} e_{\alpha_1\cdots\alpha_i} & \text{if } u = s(\alpha_i), \\ 0 & \text{if } u \neq s(\alpha_i). \end{cases}$$

By [2] string modules are indecomposable and two string modules $M(c)$ and $M(d)$ are isomorphic if and only if $c = d$ or $c = d^{-1}$. An isomorphism $M(c) \longrightarrow M(c^{-1})$ is given by sending $e_{c'}$ to $e_{c''^{-1}}$ for $c' \in \mathrm{Ld}(c)$ with $c = c'c''$. We will refer to such an isomorphism as "the isomorphism from $M(c)$ to $M(c^{-1})$".

Aside from string modules there is another type of indecomposable (finite dimensional) $\mathcal{A}$-modules, the band modules. To make it easier to describe degenerations (see section 4), we also define quasi-band modules, which are a generalization of band modules.

A quasi-band (b, m) is a map $b : \mathbb{Z} \longrightarrow Q_1 \cup Q_1^{-1}$ together with an integer $m \geq 1$ such that $b(i) = b(i + m)$ for all $i \in \mathbb{Z}$ and $b(i)b(i + 1) \cdots b(i + n)$ is a string for all $i \in \mathbb{Z}$ and all $n \geq 0$. Frequently we will just write $(b, m) = b(1) \cdots b(m)$. A quasi-band (b, m) is called a band provided (b, m') is not a quasi-band for any $0 < m' < m$. For any quasi-band (b, m) and any $\phi \in \mathrm{Aut}_k(V)$, where V is a finite dimensional k-vector space, we define the quasi-band module $M(b, m, \phi)$ in the following way. First we define an (infinite dimensional) $\mathcal{A}$-module $M(b)$ with basis $\{e_i : i \in \mathbb{Z}\}$ by

$$p \cdot e_i = \begin{cases} e_j & \text{if there is a } j \geq i \text{ such that } b(i)p^{-1} = b(i) \cdots b(j), \\ e_{j-1} & \text{if there is a } j \leq i+1 \text{ such that } pb(i+1) = b(j) \cdots b(i+1), \\ 0 & \text{otherwise,} \end{cases}$$

for any path p in Q. Note that we write $pb(i + 1) = b(j) \cdots b(i + 1)$ instead of $p = b(j) \cdots b(i)$ in order to include trivial paths. For an arrow α and and a vertex u we thus have

$$e_{i-1} \xrightarrow{\alpha} e_i \qquad \text{if } \alpha = b(i)^{-1},$$

$$e_{i-1} \xleftarrow[\alpha]{} e_i \qquad \text{if } \alpha = b(i),$$

$$1_u \cdot e_i = \begin{cases} e_i & \text{if } u = s(b(i)), \\ 0 & \text{if } u \neq s(b(i)). \end{cases}$$

We define an $\mathcal{A}$-module structure on $V \otimes_k M(b)$ by setting

$$p \cdot (v \otimes w) := v \otimes (p \cdot w)$$

for any path p in Q. Finally we set

$$M(b, m, \phi) := V \otimes_k M(b) / \operatorname{span}_k(\{v \otimes e_i - \phi(v) \otimes e_{i+m} : v \in V, i \in \mathbb{Z}\}).$$

In case $V = k$ the automorphism ϕ is given by multiplication with a $\lambda \in k^*$ and we set $M(b, m, \lambda) = M(b, m, \phi)$. We call $M(b, m, \phi)$ a band module provided (b, m) is a band and the $k[x]$-module defined by ϕ is indecomposable. By [2] any band module is indecomposable and two band modules $M(b, m, \phi)$ and $M(b', m', \phi')$ are isomorphic if and only if $m = m'$ and one of the following holds:

- There is an $i \in \mathbb{Z}$ with $b(j) = b'(i + j)$ for all $j \in \mathbb{Z}$ and ϕ and ϕ' are isomorphic as $k[x]$-modules.
- There is an $i \in \mathbb{Z}$ with $b(j) = b'(i - j)^{-1}$ for all $j \in \mathbb{Z}$ and ϕ^{-1} and ϕ' are isomorphic as $k[x]$-modules.

This motivates the definition of an equivalence relation $\sim$ for quasi-bands, defined by $(b, m) \sim (b', m')$ if $m = m'$ and one of the following holds:

- There is an $i \in \mathbb{Z}$ with $b(j) = b'(i + j)$ for all $j \in Z$.
- There is an $i \in \mathbb{Z}$ with $b(j) = b'(i - j)^{-1}$ for all $j \in Z$.

By $[(b, m)]$ we denote the equivalence class of (b, m) with respect to $\sim$.

It is shown in [2] that the finite-dimensional indecomposable $\mathcal{A}$-modules are precisely the string and band modules up to isomorphism.

For any sequence $S = (c_1, \ldots, c_l, (b_1, m_1), \cdots (b_n, m_n))$ with $l, n \geq 0$ consisting of strings $c_1, \ldots, c_l$ and quasi-bands $(b_1, m_1), \cdots, (b_m, m_n)$ the family of modules $\mathcal{F}(S) \subseteq \operatorname{mod}(\mathcal{A}, d)$ is the image of the morphism

$$\mathrm{GL}_d(k) \times (k^*)^n \longrightarrow \operatorname{mod}(\mathcal{A}, d)$$

sending $(g, \lambda_1, \ldots, \lambda_n)$ to

$$g \star (\bigoplus_{i=1}^{l} M(c_i) \oplus \bigoplus_{j=1}^{n} M(b_j, m_j, \lambda_j)),$$

where

$$d = \sum_{i=1}^{l} \dim_k M(c_i) + \sum_{j=1}^{n} \dim_k M(b_j, m_j, 1).$$

We call a subset $\mathcal{F}$ of $\mathrm{mod}(\mathcal{A}, d)$ an $\mathcal{S}$-family of strings and quasi-bands if there is a sequence S of strings and quasi-bands with $\mathcal{F} = \mathcal{F}(S)$ and we call $\mathcal{F}$ an $\mathcal{S}$-family of (strings and) bands if S is a sequence of (strings and) bands.

Note that a band module $M(b, \phi)$ with $\phi \in \mathrm{GL}_p(k) = \mathrm{Aut}_k(k^p)$ does not necessarily belong to any $\mathcal{S}$-family of strings and quasi-bands, as ϕ might not be diagonalizable. But, as the set of diagonalizable matrices in $\mathrm{GL}_p(k)$ is dense in $\mathrm{GL}_p(k)$, we see that $M(b, \phi)$ belongs to the closure of the $\mathcal{S}$-family $\mathcal{F}(b, b, \ldots, b)$, which is an $\mathcal{S}$-family of bands. Thus $\mathrm{mod}(\mathcal{A}, d)$ is a union of closures of $\mathcal{S}$-families of strings and bands. As $\mathrm{GL}_d(k)$ is irreducible, any $\mathcal{S}$-family of strings and quasi-bands is irreducible and as there are only finitely many different $\mathcal{S}$-families of strings and bands in $\mathrm{mod}(\mathcal{A}, d)$, any irreducible component of $\mathrm{mod}(\mathcal{A}, d)$ is the closure of an $\mathcal{S}$-family of strings and bands.

Let $r : \mathrm{mod}(\mathcal{A}, d) \longrightarrow \mathbb{N}$ be the function sending X to

$$r(X) = \sum_{\alpha \in Q_1} \mathrm{rank}\, X(\alpha).$$

We call $X \in \mathrm{mod}(\mathcal{A}, d)$ regular if $r(X) = d$. From the direct decomposition of X into a direct sum of string and band modules we obtain that $r(X) \leq d$ for any $X \in \mathrm{mod}(\mathcal{A}, d)$ and that X is regular if and only if X is isomorphic to a direct sum of band modules. As r is lower semi-continuous, we see that the regular elements of $\mathrm{mod}(\mathcal{A}, d)$ form an open subset of $\mathrm{mod}(\mathcal{A}, d)$.

Let $\mathcal{C}$ be an irreducible component of $\mathrm{mod}(\mathcal{A}, d)$ such that there is a regular $X \in \mathcal{C}$. We call such an irreducible component regular. We already know that there is a sequence of strings and bands S such that the closure of $\mathcal{F}(S)$ is $\mathcal{C}$. Obviously S has to be a sequence of bands. In order to determine the regular irreducible components of $\mathrm{mod}(\mathcal{A}, d)$ it suffices to solve the following problem: Given a sequence of bands S with $\mathcal{F}(S) \subseteq \mathrm{mod}(\mathcal{A}, d)$, determine whether the closure of $\mathcal{F}(S)$ is an irreducible component or not.

We apply the result on decompositions of irreducible components as presented in [3] and obtain the following: Let $S = (b_1, \ldots, b_n)$ be a sequence of bands. We set $d_i := \dim_k M(b_i, 1)$ and $d = d_1 + \ldots + d_n$. The closure of $\mathcal{F}(S)$ is an irreducible component of $\mathrm{mod}(\mathcal{A}, d)$ if and only if the following holds:

i) For all $i \neq j$, there are $X \in \mathcal{F}(b_i)$ and $Y \in \mathcal{F}(b_j)$ with $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$.

ii) The closure of $\mathcal{F}(b_i)$ is an irreducible component of $\mathrm{mod}(\mathcal{A}, d_i)$ for $i = 1, \ldots, n$.

Our goal is to characterize the conditions *i)* and *ii)* by combinatorial criteria on bands. For *i)* we have a complete solution, whereas our characterization of *ii)* only holds if the ideal I is generated by paths of length two.

Note that the result from [3] can also be applied to sequences of strings and bands in order to determine the non-regular irreducible components, but we were not able to characterize condition *ii*) for strings.

We call a pair of bands $((b,m),(c,n))$ extendable if there are $s,t \geq 1$, strings u,v,w and arrows $\alpha,\beta,\gamma,\delta$ with

$$(b,sm) = w\beta u\alpha^{-1} \text{ and } (c,tn) = w\delta^{-1}v\gamma$$

such that

$$(d,n+m) := (c,n)(b,m) := c(1)\cdots c(n)b(1)\cdots b(m)$$

is a quasi-band. Note that $l(w) > m,n$ is possible, which explains why s and t are needed. We call a pair of equivalence classes of bands (B,C) extendable, if there are bands $(b,m) \in B$ and $(c,n) \in C$ such that $((b,m),(c,n))$ is extendable.

Proposition 1.2. *Let (b,m) and (c,n) be bands. There are $X \in \mathcal{F}(b,m)$ and $Y \in \mathcal{F}(c,n)$ with $\mathrm{Ext}^1_{\mathcal{A}}(X,Y) = 0$ if and only if the pair $([(b,m)],[(c,n)])$ is not extendable.*

Note that $\mathrm{Ext}^1_{\mathcal{A}}(X,Y) = 0$ for some $X \in \mathcal{F}(b,m)$ and $Y \in \mathcal{F}(c,n)$ implies that $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$ vanishes generically on $\mathcal{F}(b,m) \times \mathcal{F}(c,n)$.

We call a band (b,m) negligible if one of the following holds:

- There are strings u,v,w,x,y, arrows $\alpha,\beta,\gamma,\delta$ and an $s \geq 1$ with

  $$(b,m) = u\gamma v\alpha^{-1} \text{ and } (b,sm) = w\beta x\alpha^{-1} = u\gamma w\delta^{-1}y$$

  such that

  $$(c,n) := u\gamma \text{ and } (d,m-n) := v\alpha^{-1}$$

  are quasi-bands.

- There is a string u that starts and ends with an arrow, a string v that starts and ends with an inverse arrow and a string w with $(b,m) = wuw^{-1}v$ such that

  $$(c,m) := wu^{-1}w^{-1}v$$

  is a quasi-band.

We call an equivalence class of bands B negligible if there is a band $(b,m) \in B$ which is negligible. One can show that (B,B) is extendable if B is negligible, but we will not use it.

Proposition 1.3. *Let (b, m) be a band with $\mathcal{F}(b, m) \subseteq \operatorname{mod}(\mathcal{A}, d)$. If the closure of $\mathcal{F}(b, m)$ is an irreducible component of $\operatorname{mod}(\mathcal{A}, d)$, then $[(b, m)]$ is not negligible.*

We do not know whether the converse holds in general, but it does in case I is generated paths of length two:

Proposition 1.4. *Assume that I is generated by a set of paths of length two and let (b, m) be a band with $\mathcal{F}(b, m) \subseteq \operatorname{mod}(\mathcal{A}, d)$. If $[(b, m)]$ is not negligible, then the closure of $\mathcal{F}(b, m)$ is an irreducible component of $\operatorname{mod}(\mathcal{A}, d)$.*

We call a sequence $S = (b_1, \ldots, b_n)$ of bands negligible, if one of the following holds:

- $[b_i]$ is negligible for some $1 \leq i \leq n$.
- $([b_i], [b_j])$ is extendable for some $1 \leq i, j \leq n$.

Our main result is the following theorem which is just a consequence from the previous propositions.

Theorem. *Let $\mathcal{A} = kQ/I$ be a string algebra and let S be a sequence of bands with $\mathcal{F}(S) \subseteq \operatorname{mod}(\mathcal{A}, d)$.*

a) If the closure of $\mathcal{F}(S)$ is an irreducible component of $\operatorname{mod}(\mathcal{A}, d)$, then S is negligible.

b) If S is negligible and I is generated by paths of length two, then the closure of $\mathcal{F}(S)$ is an irreducible component of $\operatorname{mod}(\mathcal{A}, d)$.

If I is generated by a set of paths of length two and b is a band, then $[b]$ is negligible if and only if $([b], [b])$ is extendable (see Lemma 3.1 and 3.2). From the previous theorem we thus obtain:

Corollary 1.5. *Assume that I is generated by paths of length two and let $\mathcal{F} \subseteq \operatorname{mod}(\mathcal{A}, d)$ be an $\mathcal{S}$-family of bands. The closure of $\mathcal{F}$ is an irreducible component of $\operatorname{mod}(\mathcal{A}, d)$ if and only if there are $X, Y \in \mathcal{F}$ with $\operatorname{Ext}^1_{\mathcal{A}}(X, Y) = 0$.*

Note that Corollary 1.5 is not true if I is not generated by paths of length two. Indeed, consider the algebra $\Lambda = k[\alpha, \beta]/(\alpha^3, \beta^3, \alpha\beta)$ and the band $\alpha^{-1}\beta$. The closure of $\mathcal{F}(\alpha^{-1}\beta)$ is an irreducible component of $\operatorname{mod}(\Lambda, 2)$, as there are no other $\mathcal{S}$-families of band modules in $\operatorname{mod}(\Lambda, 2)$. On the other hand, $\operatorname{Ext}^1_{\Lambda}(X, Y)$ does not vanish for any $X, Y \in \mathcal{F}(\alpha^{-1}\beta)$, as one can

easily construct a short exact sequence $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$ for some $Z \in \mathcal{F}(\alpha^{-2}\beta^2)$.

If I is generated by paths of length two, there is a simple formula for the dimension of a regular irreducible component: We call a vertex $u \in Q_0$ gentle (w.r.t. I) if it satisfies the following:

- For any arrow α of Q with $s(\alpha) = u$ there is a most one arrow β with $s(\alpha) = t(\beta)$ and $\alpha\beta \in I$.

- For any arrow β of Q with $t(\beta) = u$ there is a most one arrow α with $s(\alpha) = t(\beta)$ and $\alpha\beta \in I$.

We call $\mathcal{A}$ a gentle algebra if any vertex of Q is gentle and the ideal I is generated by paths of length two.

Proposition 1.6. *Assume that I is generated by paths of length two and let S be a sequence of bands such that the closure $\overline{\mathcal{F}(S)}$ is an irreducible component of* $\mathrm{mod}(\mathcal{A}, d)$*. The dimension of $\overline{\mathcal{F}(S)}$ is given by the formula*

$$\dim \overline{\mathcal{F}(S)} = d^2 - \sum_{\substack{u \in Q_0 \\ u \text{ non-gentle}}} \dim_k \mathrm{Hom}_{\mathcal{A}}(X, M(1_u)) \dim_k \mathrm{Hom}_{\mathcal{A}}(M(1_u), X),$$

for any $X \in \mathcal{F}(S)$. In particular, the dimension of any regular irreducible component of $\mathrm{mod}(\mathcal{A}, d)$ *is d^2 provided $\mathcal{A}$ is a gentle algebra.*

The paper is organized as follows. In section 2 we recall results on homomorphisms between representations of string algebras from [5]. In section 3 we prove Proposition 1.4 and the dimension formula Proposition 1.6. Section 4 is devoted to explicit inclusions among closures of $\mathcal{S}$-families of bands and the proof of Proposition 1.3. In section 5 we study extensions and prove Proposition 1.2.

2 Homomorphisms

We show that $\mathcal{S}$-families of band modules can be separated by hom-conditions using string modules (see Proposition 2.5), a result we need for the proof of Proposition 1.4. We recall a basis of homomorphism spaces between representations of string algebras worked out in [5].

2.1 Substring morphisms for a string

Let c be a string. A substring of c is a triple (c_1, c_2, c_3) of strings with $c = c_1c_2c_3$ satisfying the following:

- c_1 is either trivial or it starts with an inverse arrow ($c_1 = c_1'\alpha^{-1}$).
- c_3 is either trivial or it ends with an arrow ($c_3 = \alpha c_3'$).

By $\mathrm{sub}(c)$ we denote the set of substrings of c.

For each $(c_1, c_2, c_3) \in \mathrm{sub}(c)$ we define the homomorphism

$$\iota_{c_2,(c_1,c_2,c_3)} : M(c_2) \longrightarrow M(c)$$

by sending e_d to e_{c_1d} for $d \in \mathrm{Ld}(c_2)$. We call such a morphism a substring morphism. For a string d we set

$$\mathrm{sub}(d, c) := \{(c_1, c_2, c_3) \in \mathrm{sub}(c) : c_2 \in \{d, d^{-1}\}\}.$$

2.2 Factorstring morphisms for a string

Let c be a string. A factorstring of c is a triple (c_1, c_2, c_3) of strings with $c = c_1c_2c_3$ satisfying the following:

- c_1 is either trivial or it starts with an arrow ($c_1 = c_1'\alpha$).
- c_3 is either trivial or it ends with an inverse arrow ($c_3 = \alpha^{-1}c_3'$).

By $\mathrm{fac}(c)$ we denote the set of substrings of c.

For each $(c_1, c_2, c_3) \in \mathrm{fac}(c)$ we define the homomorphism

$$\pi_{c_2,(c_1,c_2,c_3)} : M(c) \longrightarrow M(c_2)$$

by sending e_d to

$$\begin{cases} e_{d'} & \text{if } d = c_1d' \in \mathrm{Ld}(c_1c_2) \\ 0 & \text{otherwise.} \end{cases}$$

We call such a morphism a factorstring morphism. For a string d we set

$$\mathrm{fac}(d, c) := \{(c_1, c_2, c_3) \in \mathrm{fac}(c) : c_2 \in \{d, d^{-1}\}\}.$$

2.3 Winding and unwinding morphisms

Let (b, m) be a quasi-band. For $s \geq 1$ and $\lambda \in k^*$ we define the winding morphism

$$w_{(b,m),s,\lambda} : M(b, sm, \lambda^s) \longrightarrow M(b, m, \lambda)$$

by sending $\overline{1 \otimes e_i}$ to $\overline{1 \otimes e_i}$ for $i = 0, \dots, sm - 1$.

Dually, the unwinding morphism

$$u_{(b,m),s,\lambda} : M(b, m, \lambda) \longrightarrow M(b, sm, \lambda^s),$$

sends $\overline{1 \otimes e_i}$ to

$$\sum_{j=0}^{s-1} \lambda^j \overline{1 \otimes e_{i+jm}}$$

for $i = 0, \dots, m - 1$.

2.4 Substring morphisms for a quasi-band

Let (b, m) be a quasi-band and c a string. We define the set

$$\begin{aligned} \mathrm{sub}_1(c, (b, m)) \quad := \quad & \{1 \leq i \leq m : b(i) \cdots b(i + l(c)) = b(i)c, \\ & b(i)^{-1}, b(i + l(c) + 1) \in Q_1\} \end{aligned}$$

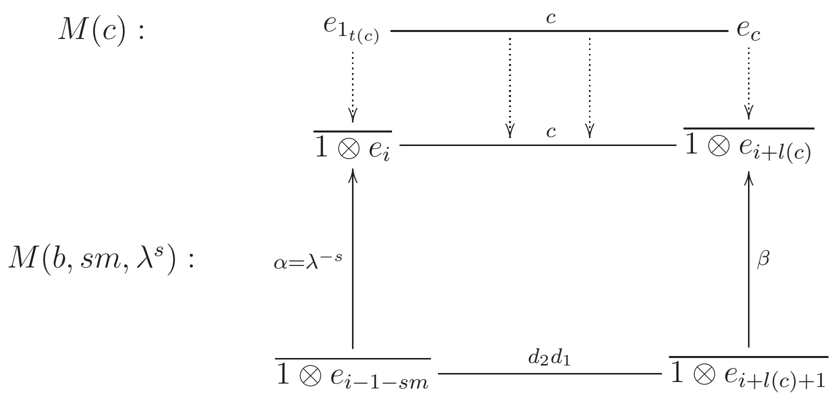
Note that we write $b(i) \cdots b(i+l(c)) = b(i)c$ instead of $b(i+1) \cdots b(i+l(c)) = c$ in order to include the case $l(c) = 0$. For any $i \in \mathrm{sub}_1(c, (b, m))$ and any $\lambda \in k^*$ we define a morphism

$$\iota_{i,c,(b,m),\lambda} : M(c) \longrightarrow M(b, m, \lambda)$$

as a composition

$$M(c) \xrightarrow{f} M(b, sm, \lambda^s) \xrightarrow{w_{(b,m),s,\lambda}} M(b, m, \lambda)$$

for some integer $s \geq 1$ chosen in such a way that $(b, sm) = d_1\alpha^{-1}c\beta d_2$ for some arrows α, β and some strings d_1, d_2 with $l(d_1) = i - 1$, where the morphism f sends e_d to $\overline{1 \otimes e_{i+l(d)}}$ for $d \in \mathrm{Ld}(c)$.



Note that $\iota_{i,c,(b,m),\lambda}$ does not depend on the choice of s.

We set $\mathrm{sub}_{-1}(c,(b,m)) := \mathrm{sub}(c^{-1},(b,m))$. Note that $\mathrm{sub}_{-1}(c,(b,m)) = \mathrm{sub}_1(c,(b,m))$ if c is trivial. For each $i \in \mathrm{sub}_{-1}(c,(b,m))$ we define a morphism

$$\iota_{i,c,(b,m),\lambda} : M(c) \longrightarrow M(b,m,\lambda).$$

as the composition

$$M(c) \xrightarrow{\sim} M(c^{-1}) \xrightarrow{\iota_{i,c^{-1},(b,m),\lambda}} M(b,m,\lambda),$$

where the first morphism is the isomorphism from $M(c)$ to $M(c^{-1})$.

We have thus defined morphisms $\iota_{i,c,(b,m),\lambda}$, called substring morphisms, for any $\lambda \in k^*$ and any

$$i \in \mathrm{sub}(c,(b,m)) := \mathrm{sub}_1(c,(b,m)) \cup \mathrm{sub}_{-1}(c,(b,m)).$$

Whereas substring morphisms for strings are always injective, substring morphisms for quasi-bands are not necessarily.

Dually we define the factorstring morphisms for quasi-bands:

2.5 Factorstring morphisms for a quasi-band

Let (b,m) be a quasi-band and c a string. We define the set

$$\begin{aligned}\mathrm{fac}_1(c,(b,m)) := \{&1 \le i \le m : b(i)\cdots b(i+l(c)) = b(i)c,\\ &b(i), b(i+l(c)+1)^{-1} \in Q_1\}\end{aligned}$$

For any $i \in \mathrm{fac}_1(c,(b,m))$ and any $\lambda \in k^*$ we define a morphism

$$\pi_{i,c,(b,m),\lambda} : M(b,m,\lambda) \longrightarrow M(c)$$

as a composition

$$M(b,m,\lambda) \xrightarrow{u_{(b,m),s,\lambda}} M(b,sm,\lambda^s) \xrightarrow{f} M(c)$$

for some integer $s \geq 1$ chosen in such a way that $(b,sm) = d_1\alpha c\beta^{-1}d_2$ for some arrows α,β and some strings d_1,d_2 with $l(d_1) = i-1$, where f is the morphism that sends $\overline{1\otimes e_j}$ to 0 if either $0 \leq j < i$ or $i + l(c) < j < sm$, and $\overline{1\otimes e_j}$ to e_d if $i \leq j \leq i + l(c)$, where d is the leftdivisor of c of length $j-i$.

We set $\mathrm{fac}_{-1}(c,(b,m)) := \mathrm{fac}(c^{-1},(b,m))$. For each $i \in \mathrm{fac}_{-1}(c,(b,m))$ we obtain a morphism

$$\pi_{i,c,(b,m),\lambda} : M(b,m,\lambda) \longrightarrow M(c)$$

by identifying $M(c)$ and $M(c^{-1})$ just as above.

We have thus defined morphisms $\pi_{i,c,(b,m),\lambda}$, called factorstring morphisms, for any $\lambda \in k^*$ and any

$$i \in \mathrm{fac}(c,(b,m)) := \mathrm{fac}_1(c,(b,m)) \cup \mathrm{fac}_{-1}(c,(b,m)).$$

2.6 Morphisms between string and band modules

In this section we will frequently use the abbreviation

$$[X,Y] := \dim_k \mathrm{Hom}_{\mathcal{A}}(X,Y)$$

for $\mathcal{A}$-modules X,Y. The following three propositions are reformulations of results from [5].

Proposition 2.1. *Let $M(b,m,\lambda)$ be a band module and $M(c)$ a string module. The morphisms*

$$\iota_{d,x} \circ \pi_{i,d,(b,m),\lambda} : M(b,m,\lambda) \longrightarrow M(c),$$

where d is a string of length at most $l(c)$, $x \in \mathrm{sub}(d,c)$ and $i \in \mathrm{fac}(d,(b,m))$, form a basis of $\mathrm{Hom}_{\mathcal{A}}(M(b,m,\lambda),M(c))$. In particular,

$$[M(b,m,\lambda),M(c)] = \sum_{d\in\mathcal{W},l(d)\leq l(c)} \sharp\,\mathrm{fac}(d,(b,m))\sharp\,\mathrm{sub}(d,c).$$

Proposition 2.2. *Let $M(b,m,\lambda)$ be a band module and $M(c)$ a string module. The morphisms*

$$\iota_{i,d,(b,m),\lambda} \circ \pi_{d,x} : M(c) \longrightarrow M(b,m,\lambda),$$

where d is a string of length at most $l(c)$, $x \in \mathrm{fac}(d, c)$ and $i \in \mathrm{sub}(d, (b, m))$, form a basis of $\mathrm{Hom}_{\mathcal{A}}(M(c), M(b, m, \lambda))$. *In particular,*

$$[M(c), M(b, m, \lambda)] = \sum_{d \in \mathcal{W}, l(d) \leq l(c)} \sharp\, \mathrm{fac}(d, c) \sharp\, \mathrm{sub}(d, (b, m)).$$

Proposition 2.3. *Let $M(b, m, \lambda)$ and $M(c, n, \mu)$ be band modules. The morphisms*

$$\iota_{j,d,(c,n),\mu} \circ \pi_{i,d,(b,m),\lambda} : M(b, m, \lambda) \longrightarrow M(c, n, \mu),$$

where d is a string, $j \in \mathrm{sub}(d, (c, n))$ and $i \in \mathrm{fac}(d, (b, m))$ (together with an isomorphism in case $M(b, m, \lambda)$ and $M(c, n, \mu)$ are isomorphic) form a basis of $\mathrm{Hom}_{\mathcal{A}}(M(b, m, \lambda), M(c, n, \mu))$.

As an example, we present a result which we will need in section 5.

Lemma 2.4. *Let $M(b, m, \lambda)$ and $M(c, n, \mu)$ be band modules, d a string, $j \in \mathrm{sub}(d, (c, n))$ and $i \in \mathrm{fac}(d, (b, m))$. The morphism*

$$\iota_{j,d,(c,n),\mu} \circ \pi_{i,d,(b,m),\lambda} : M(b, m, \lambda) \longrightarrow M(c, n, \mu)$$

is injective if $m \leq l(d) < n+m$ and $m < n$, and it is surjective if $n \leq l(d) < m + n$ and $n < m$.

Note that Lemma 2.4 may become wrong if we drop the condition $l(d) < m + n$.

Proof. We may assume that $j = n$ and $i = m$. Up to duality it suffices to show that the morphism

$$f := \iota_{j,d,(c,n),\mu} \circ \pi_{i,d,(b,m),\lambda}$$

is injective if $m \leq l(d) < n+m$ and $m < n$. Let A be the matrix of f with respect to the bases $\overline{1 \otimes e_0}, \ldots, \overline{1 \otimes e_{m-1}}$ of $M(b, m, \lambda)$ and $\overline{1 \otimes e_0}, \ldots, \overline{1 \otimes e_{n-1}}$ of $M(c, n, \mu)$. To show that f is injective, we list the possible forms of A depending on the relation between n, m and $l(d)$:
If $l(d) < n$, then A is of the form

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

where $A_1 \in \mathrm{Mat}(m \times m, k)$ is the identity matrix and $A_2 \in \mathrm{Mat}(n-m \times m, k)$. From now on we assume that $n \leq l(d)$. If $2m \leq n$, then A is of the form

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix},$$

where $A_1, A_2 \in \mathrm{Mat}(m \times m, k)$, $A_3 \in \mathrm{Mat}(n-2m \times m, k)$ and $A_2 = \lambda \cdot 1_{m\times m}$ is a multiple of the identity matrix. Finally, we assume that $n < 2m$. We decompose $A = B + C$, where

$$B = \begin{pmatrix} 1_{n-m\times n-m} & 0_{n-m\times 2m-n} \\ 0_{2m-n\times n-m} & 1_{2m-n\times 2m-n} \\ \lambda\cdot 1_{n-m\times n-m} & 0_{n-m\times 2m-n} \end{pmatrix},$$

$$C = \begin{pmatrix} 0_{2m-n\times n-m} & C_1 \\ C_2 & 0_{n-m\times 2m-n} \\ 0_{n-m\times n-m} & 0_{n-m\times 2m-n} \end{pmatrix}$$

and C_1 and C_2 are diagonal matrices. Note that the sizes of the blocks in B and C are not necessarily the same. Now we see that A is of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix},$$

where

- $A_{11}, A_{31} \in \mathrm{Mat}(n-m \times n-m, k)$,
- $A_{12}, A_{32} \in \mathrm{Mat}(n-m \times 2m-n)$,
- $A_{21} \in \mathrm{Mat}(2m-n \times n-m, k)$,
- $A_{22} \in \mathrm{Mat}(2m-n \times 2m-n)$,
- A_{22} is upper triangular and all its entries on the diagonal are 1,
- $A_{31} = \lambda \cdot 1_{n-m\times n-m}$ and
- A_{32} is zero.

$\square$

The following proposition shows that the functions

$$[M(c), -], [-, M(c)] : \mathrm{mod}(\mathcal{A}, d) \longrightarrow \mathbb{N}$$

for $c \in \mathcal{W}$ separate $\mathcal{S}$-families of band modules.

Proposition 2.5. *Let $S = (b_1, \ldots, b_m)$ and $T = (c_1, \ldots, c_n)$ be sequences of bands with $\mathcal{F}(S), \mathcal{F}(T) \subseteq \mathrm{mod}(\mathcal{A}, d)$. If*

$$[M(c), X] = [M(c), Y] \text{ and } [X, M(c)] = [Y, M(c)]$$

for any string c, $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$, then

$$\mathcal{F}(S) = \mathcal{F}(T).$$

We do not claim that Proposition 2.5 holds for sequences of quasi-bands.

Let X, Y be $\mathcal{A}$-modules. Clearly the dimension vectors of X and Y coincide (i.e. $\dim_k \operatorname{im} X(1_u) = \dim_k \operatorname{im} Y(1_u)$ for all $u \in Q_0$) if and only if $[P, X] = [P, Y]$ for any projective $\mathcal{A}$-module P if and only if $[X, J] = [Y, J]$ for any injective $\mathcal{A}$-module J. Recall from [1], that

$$[U, X] - [X, \tau U] = [P_0, X] - [P_1, X],$$

where $P_1 \longrightarrow P_0 \longrightarrow U \longrightarrow 0$ is a minimal projective presentation of an $\mathcal{A}$-module U and τ denotes the Auslander-Reiten translation. Dually, if $0 \longrightarrow U \longrightarrow J_0 \longrightarrow J_1$ is a minimal injective copresentation of U, then

$$[X, U] - [\tau^{-}U, X] = [X, J_0] - [X, J_1].$$

As the Auslander-Reiten translate of a string module is either 0 or a string module (see [2]) and as all projective and injective $\mathcal{A}$-modules are string modules, we obtain the following corollary.

Corollary 2.6. *Let S and T be sequences of bands with $\mathcal{F}(S), \mathcal{F}(T) \subseteq \operatorname{mod}(\mathcal{A}, d)$ and let $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$. The following are equivalent:*

i) $\mathcal{F}(S) = \mathcal{F}(T)$

ii) $[M(c), X] = [M(c), Y]$ *for any string c.*

iii) $[X, M(c)] = [Y, M(c)]$ *for any string c.*

Before we can prove Proposition 2.5 we need some additional definitions and a technical lemma. For any non-trivial string c and any quasi-band (b, m) we set

- $\operatorname{part}_1(c, (b, m)) := \{1 \leq i \leq m : b(i) \cdots b(i + l(c) - 1) = c\}$,
- $\operatorname{part}_{-1}(c, (b, m)) := \operatorname{part}_1(c^{-1}, (b, m))$ and
- $\operatorname{part}(c, (b, m)) := \operatorname{part}_1(c, (b, m)) \cup \operatorname{part}_{-1}(c, (b, m))$.

We extend the definition of $\operatorname{sub}(c, -), \operatorname{fac}(c, -)$ for a string c and $\operatorname{part}(c, -)$ for a non-trivial string c to sequences of quasi-bands instead of a single quasi-band. For a sequence $S = (b_1, \ldots, b_n)$ of bands, we set

- $\operatorname{part}(c, S) := \bigcup_{i=1}^{n} (\operatorname{part}(c, b_i) \times \{i\}) \subseteq \mathbb{N} \times \mathbb{N}$
- $\operatorname{sub}(c, S) := \bigcup_{i=1}^{n} (\operatorname{sub}(c, b_i) \times \{i\}) \subseteq \mathbb{N} \times \mathbb{N}$

- $\operatorname{fac}(c,S) := \bigcup_{i=1}^{n} (\operatorname{fac}(c,b_i) \times \{i\}) \subseteq \mathbb{N} \times \mathbb{N}$

Moreover, we define

$$[c,S] := \sum_{d \in \mathcal{W}, l(d) \leq l(c)} \sharp \operatorname{fac}(d,c) \sharp \operatorname{sub}(d,S)$$

and

$$[S,c] := \sum_{d \in \mathcal{W}, l(d) \leq l(c)} \sharp \operatorname{fac}(d,S) \sharp \operatorname{sub}(d,c).$$

As direct consequence of Proposition 2.1 and Proposition 2.2 we obtain

Corollary 2.7. *Let S be a sequence of bands and $X \in \mathcal{F}(S)$. For any string c we have*

$$[c,S] = [M(c),X] \text{ and } [S,c] = [X,M(c)].$$

We come to the technical lemma.

Lemma 2.8. *Let S and T be sequences of bands with $\operatorname{rank} X(\alpha) = \operatorname{rank} Y(\alpha)$ for any arrow α, $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$ and let $N \in \mathbb{N}$. If $[c,S] = [c,T]$ and $[S,c] = [T,c]$ for any string c of length at most N, then $\sharp \operatorname{part}(d,S) = \sharp \operatorname{part}(d,T)$ for any non-trivial string d of length at most $N+2$.*

Proof. It follows from Corollary 2.7 that $\sharp \operatorname{sub}(c,S) = \sharp \operatorname{sub}(c,T)$ and $\sharp \operatorname{fac}(c,S) = \sharp \operatorname{fac}(c,T)$ for any string c of length at most N. We will prove that $\sharp \operatorname{part}(d,S) = \sharp \operatorname{part}(d,T)$ for $1 \leq l(d) \leq N+2$ by induction on the length of d. If $l(d) = 1$, we may assume that d is an arrow. Let $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$. We have

$$\sharp \operatorname{part}(d,S) = \operatorname{rank} X(d) = \operatorname{rank} Y(d) = \sharp \operatorname{part}(d,T).$$

If $N+2 \geq l(d) > 1$, then d is of the form $d = d_1 c d_2$ for a (possibly trivial) string c of length at most N and strings d_1, d_2 of length one. We assume that $\sharp \operatorname{part}(d,S) \neq \sharp \operatorname{part}(d,T)$. By exchanging S and T we can assume that $\sharp \operatorname{part}(d,S) > \sharp \operatorname{part}(d,T)$. By the induction hypothesis we know that $\sharp \operatorname{part}(d_1 c,S) = \sharp \operatorname{part}(d_1 c,T)$ and thus

$$\begin{aligned} \sharp \operatorname{part}(d_1 c,T) &= \sharp \operatorname{part}(d_1 c,S) \\ &\geq \sharp \operatorname{part}(d_1 c d_2,S) \\ &> \sharp \operatorname{part}(d_1 c d_2,T) \end{aligned}$$

This shows that $\operatorname{part}(d_1 c,T) - \operatorname{part}(d_1 c d_2,T)$ is non-empty, which implies that there must be a string $d_3 \neq d_2$ of length one such that $d_1 c d_3$ is a string.

As d_1c is non-trivial, there is a most one arrow α such that $d_1c\alpha$ is a string and at most one arrow β such that $d_1c\beta^{-1}$ is a string. Consequently, such arrows α and β exist and satisfy $\{\alpha, \beta^{-1}\} = \{d_2, d_3\}$. We obtain

$$\begin{aligned}
\sharp\,\mathrm{part}(d_1cd_2, S) + \sharp\,\mathrm{part}(d_1cd_3, S) &= \sharp\,\mathrm{part}(d_1c, S) \\
&= \sharp\,\mathrm{part}(d_1c, T) \\
&= \sharp\,\mathrm{part}(d_1cd_2, T) + \sharp\,\mathrm{part}(d_1cd_3, T)
\end{aligned}$$

and thus $\sharp\,\mathrm{part}(d_1cd_3, S) \neq \sharp\,\mathrm{part}(d_1cd_3, T)$. If d_1 is an arrow, then $\sharp\,\mathrm{fac}(c, S) \neq \sharp\,\mathrm{fac}(c, T)$ and if d_1^{-1} is an arrow, then $\sharp\,\mathrm{sub}(c, S) \neq \sharp\,\mathrm{sub}(c, T)$. This gives a contradiction in any case. $\square$

Proof of Proposition 2.5. By the definition of $\mathcal{S}$-families of band modules it suffices to show that

$$\sharp\{1 \leq i \leq m : [b_i] = [b]\} = \sharp\{1 \leq i \leq n : [c_i] = [b]\}$$

for any band b. We first show that $\mathrm{part}(d, S) = \mathrm{part}(d, T)$ for any non-trivial string d. In order to apply Lemma 2.8, we need to show that $\mathrm{rank}\, X(\alpha) = \mathrm{rank}\, Y(\alpha)$ for any arrow α, $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$. Let α be an arrow. Let p and q be the paths of maximal length such that $q\alpha$ and αp^{-1} are strings.

Note that $M(q\alpha p^{-1}) = P_{s(\alpha)}$, where $P_{s(\alpha)}$ is the indecomposable projective module corresponding to the vertex $s(\alpha)$, and that $M(p^{-1})$ is the cokernel of the morphism $P_{t(\alpha)} \longrightarrow P_{s(\alpha)}$. Applying $\mathrm{Hom}_{\mathcal{A}}(-, X)$, we obtain the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M(p^{-1}), X) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(P_{s(\alpha)}, X) \arrow[r] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathcal{A}}(P_{t(\alpha)}, X) \arrow[d, "\sim"] \\
& & \mathrm{im}\, X(1_{s(\alpha)}) \arrow[r, "X(\alpha)"'] & \mathrm{im}\, X(1_{t(\alpha)})
\end{tikzcd}$$

which shows that $\mathrm{rank}\, X(\alpha) = [P_{s(\alpha)}, X] - [M(p^{-1}), X]$ and thus

$$\begin{aligned}
\mathrm{rank}\, X(\alpha) &= [P_{s(\alpha)}, X] - [M(p^{-1}), X] \\
&= [P_{s(\alpha)}, Y] - [M(p^{-1}), Y] \\
&= \mathrm{rank}\, Y(\alpha).
\end{aligned}$$

Applying Lemma 2.8, we obtain $\sharp\,\mathrm{part}(d, S) = \sharp\,\mathrm{part}(d, T)$ for any non-trivial string d, as desired.

Let $b = (b, l)$ be a band. For any $k \geq 1$ we define the string

$$d_k := (b, kl) = b(1)b(2)\cdots b(kl).$$

Clearly $\sharp\,\mathrm{part}(d_k,(b,l)) = 1$ for any $k \geq 1$ and if (c,j) is a band with $(c,j) \nsim (b,l)$, then

$$\sharp\,\mathrm{part}(d_k,(c,j)) = 0$$

for sufficiently large k. We choose $K \in \mathbb{N}$, such that

$$\sharp\,\mathrm{part}(d_K,x) = \begin{cases} 1 & \text{if } x \sim b \\ 0 & \text{otherwise.} \end{cases}$$

for any $x \in \{b_1,\ldots,b_m,c_1,\ldots,c_n\}$. We obtain the desired equality

$$\begin{aligned} \sharp\{1 \leq i \leq m : [b_i] = [b]\} &= \sharp\,\mathrm{part}(d_K,S) \\ &= \sharp\,\mathrm{part}(d_K,T) \\ &= \sharp\{1 \leq i \leq n : [c_i] = [b]\} \end{aligned}$$

□

3 Proof of Proposition 1.4 and 1.6

In this section we assume that $\mathcal{A} = kQ/I$ is a string algebra such that I is generated by a set of paths of length two. For the proof of Proposition 1.4 we need the following characterization of negligibility.

Lemma 3.1. *The equivalence class of a band $b = (b,m)$ is negligible if and only if the following holds: There are a string c and arrows $\alpha,\beta,\gamma,\delta$ such that the sets $\mathrm{part}(\alpha^{-1}c\beta,b)$ and $\mathrm{part}(\gamma c\delta^{-1},b)$ are non-empty and $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ are strings.*

Proof. If (b,m) is negligible, one can find a string c and arrows $\alpha,\beta,\gamma,\delta$ such that the sets $\mathrm{part}(\alpha^{-1}c\beta,b)$ and $\mathrm{part}(\gamma c\delta^{-1},b)$ are non-empty and $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ are strings, by a simple case-by-case analysis which we omit. Indeed, the choice $c = w$ will work in both cases.

We now assume that there are a string c and arrows $\alpha,\beta,\gamma,\delta$ such that the sets $\mathrm{part}(\alpha^{-1}c\beta,b)$ and $\mathrm{part}(\gamma c\delta^{-1},b)$ are non-empty and $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ are strings, and we want to show that (b,m) is negligible. Up to replacing (b,m) by an equivalent band, we may assume that $m \in \mathrm{part}_1(\alpha^{-1}c\beta,(b,m))$. We choose $n \in \mathrm{part}(\gamma c\delta^{-1},(b,m))$. There are two cases to consider:

- $n \in \mathrm{part}_1(\gamma c\delta^{-1},(b,m))$
- $n \in \mathrm{part}_{-1}(\gamma c\delta^{-1},(b,m))$

If $n \in \mathrm{part}_1(\gamma c\delta^{-1}, (b, m))$, we set

$$w = c, u = b(1) \cdots b(n-1), v = b(n+1) \cdots b(m-1).$$

We have

$$(b, m) = u\gamma v\alpha^{-1} \text{ and } (b, sm) = w\beta x\alpha^{-1} = u\gamma w\delta^{-1}y$$

for an integer $s \geq 1$ and strings x, y. From now on we use that I is generated by paths of length two. As $\gamma c\delta^{-1}$ is a string, we see that $\gamma b(1)$ is a string and thus γu is a string as well. As γu and $u\gamma$ are both strings, we obtain that $u\gamma$ is a quasi-band. Similarly one can show that $v\alpha^{-1}$ is a quasi-band.

We now assume that $n \in \mathrm{part}_{-1}(\gamma c\delta^{-1}, (b, m))$. By the definition of the sets part_1 and part_{-1} we have

$$b(i) = b(n + l(c) + 1 - i)^{-1}$$

for $1 \leq i \leq l(c)$. Note that $l(c) < n$, as otherwise

$$\beta = b(l(c) + 1) = b(n)^{-1} = \delta^{-1}.$$

Similarly we obtain that $n + l(c) < m$. Thus the band (b, m) is of the form

$$(b, m) = cuc^{-1}v$$

for some non-trivial strings u and v of length $l(u) = n - l(c)$ and $l(v) = m - n - l(c)$. As $\beta = b(l(c) + 1)$ and $\delta = b(n)$, we see that u starts and ends with an arrow. Similarly we obtain that v starts and ends with an inverse arrow. In order to show that (b, m) is negligible, it remains to prove that

$$(d, m) := cu^{-1}c^{-1}v$$

is a quasi-band. As I is generated by paths of length two, it suffices to show that

$$u^{-1}c^{-1}v \text{ and } vcu^{-1}$$

are strings. We show that vcu^{-1} is a string. We decompose $v = v'\alpha^{-1}$ and $u = u'\delta$. As

$$v'\alpha^{-1} \text{ and } \alpha^{-1}c\delta^{-1} \text{ and } \delta^{-1}(u')^{-1}$$

are strings, we obtain that

$$vcu^{-1} = v'\alpha^{-1}c\delta^{-1}(u')^{-1}$$

is a string. Similarly one can show that $u^{-1}c^{-1}v$ is a string. $\square$

Proof of Proposition 1.4. Let $b = (b, m)$ be a band such that the closure of $\mathcal{F}(b) \subseteq \operatorname{mod}(\mathcal{A}, d)$ is not an irreducible component of $\operatorname{mod}(\mathcal{A}, d)$. We assume that (b, m) is not negligible and want to obtain a contradiction.

As $\mathcal{F}(b)$ is irreducible it must be contained in a irreducible component $\mathcal{C}$, which is regular as it contains $\mathcal{F}(b)$. Thus there is a sequence $S = (b_1, \dots, b_n)$ of bands such that the closure of $\mathcal{F}(S)$ is $\mathcal{C}$. As the function $X \mapsto \operatorname{rank} X(\alpha)$ is lower semi-continuous on $\operatorname{mod}(\mathcal{A}, d)$, we see that $\operatorname{rank} X(\alpha) \leq \operatorname{rank} Y(\alpha)$ for any arrow α, any $X \in \mathcal{F}(b)$ and any $Y \in \mathcal{F}(S)$. On the other hand,

$$\sum_{\alpha \in Q_1} \operatorname{rank} X(\alpha) = r(X) = d = r(Y) = \sum_{\alpha \in Q_1} \operatorname{rank} Y(\alpha)$$

and thus

$$\sharp \operatorname{part}(\alpha, b) = \operatorname{rank} X(\alpha) = \operatorname{rank} Y(\alpha) = \sharp \operatorname{part}(\alpha, S).$$

For any string c the functions $[M(c), -]$ and $[-, M(c)]$ from $\operatorname{mod}(\mathcal{A}, d)$ to $\mathbb{N}$ are upper semi-continuous and thus

$$[c, b] \geq [c, S] \text{ and } [b, c] \geq [S, c].$$

As we know by Proposition 2.5 that strings separate $\mathcal{S}$-families of bands, there is a string c of minimal length with the property that $[c, b] > [c, S]$ or $[b, c] > [S, c]$. We only examine the case $[c, b] > [c, S]$, as the other case is treated similarly. It follows from Corollary 2.7 that $\sharp \operatorname{sub}(c, b) > \sharp \operatorname{sub}(c, S)$. Thus there are arrows α, β such that $\alpha^{-1} c \beta$ is a string and

$$\sharp \operatorname{part}(\alpha^{-1} c \beta, b) > \sharp \operatorname{part}(\alpha^{-1} c \beta, S).$$

By Lemma 2.8 $\sharp \operatorname{part}(c\beta, b) = \sharp \operatorname{part}(c\beta, S)$ and therefore there is an arrow γ such that $\gamma c \beta$ is a string and

$$\begin{aligned} \sharp \operatorname{part}(\alpha^{-1} c \beta, b) + \sharp \operatorname{part}(\gamma c \beta, b) &= \sharp \operatorname{part}(c\beta, b) \\ &= \sharp \operatorname{part}(c\beta, S) \\ &= \sharp \operatorname{part}(\alpha^{-1} c \beta, S) + \sharp \operatorname{part}(\gamma c \beta, S). \end{aligned}$$

In particular $\sharp \operatorname{part}(\gamma c \beta, b) < \sharp \operatorname{part}(\gamma c \beta, S)$. Similarly we find an arrow δ such that $\alpha^{-1} c \delta^{-1}$ is a string and $\sharp \operatorname{part}(\alpha^{-1} c \delta^{-1}, b) < \sharp \operatorname{part}(\alpha^{-1} c \delta^{-1}, S)$. We obtain

$$\sharp \operatorname{part}(\gamma c, b) - \sharp \operatorname{part}(\gamma c \beta, b) > \sharp \operatorname{part}(\gamma c, S) - \sharp \operatorname{part}(\gamma c \beta, S) \geq 0.$$

Hence the word $\gamma c \delta^{-1}$ has to be a string and satisfies

$$\sharp \operatorname{part}(\gamma c \delta^{-1}, b) = \sharp \operatorname{part}(\gamma c, b) - \sharp \operatorname{part}(\gamma c \beta, b) > 0.$$

From the characterization of negligibility in Lemma 3.1 we obtain that $[b]$ is not negligible. $\square$

For the proof of the dimension formula Proposition 1.6 we need another lemma.

Lemma 3.2. *Let (b, m) and (c, n) be bands. The pair $[(b, m), (c, n)]$ is extendable if and only if the following holds: There are a string d and arrows $\alpha, \beta, \gamma, \delta$ such that the sets* $\mathrm{part}(\alpha^{-1}d\beta, (b, m))$ *and* $\mathrm{part}(\gamma d\delta^{-1}, (c, n))$ *are non-empty and $\alpha^{-1}d\delta^{-1}$ and $\gamma d\beta$ are strings.*

Proof. Set $w = d$ in the definition of extendability and observe that $(d, n+m)$ is a quasi-band if and only if $\alpha^{-1}d\delta^{-1}$ and $\gamma d\beta$ are strings. □

Proof of Proposition 1.6. Let $S = (b_1, \ldots, b_n)$ be a sequence of bands such that the closure of $\mathcal{F}(S)$ is an irreducible component in $\mathrm{mod}(\mathcal{A}, d)$. From the first part of the main theorem we know that

- $([b_i], [b_j])$ is not extendable for $i \neq j$ and
- $[b_i]$ is not negligible for all i.

Let $M = M(b_1, \lambda_1) \oplus \cdots \oplus M(b_n, \lambda_n) \in \mathcal{F}(S)$ such that $M(b_1, \lambda_1), \ldots, M(b_n, \lambda_n)$ are pairwise non-isomorphic. The dimension of $\overline{\mathcal{F}(S)}$ is given by the formula

$$\dim \overline{\mathcal{F}(S)} = d^2 + n - [M, M],$$

as $\mathcal{F}(S)$ is an n-parameter family orbits of dimension $d^2 - [M, M]$. Let N be the set of all tuples (i, j, k, l, c) consisting of integers i, j, k, l and a string c such that $i \in \mathrm{fac}(c, b_k)$ and $j \in \mathrm{sub}(c, b_l)$. By Proposition 2.3 a basis of the space $\mathrm{Hom}_{\mathcal{A}}(M, M)$ is given by

- $\sharp N$ morphisms corresponding tuples $(i, j, k, l, c) \in N$
- n morphisms corresponding to the identities on M_i for $i = 1, \ldots, n$.

We thus obtain

$$\dim \overline{\mathcal{F}(S)} = d^2 - \sharp N$$

It remains to show that the cardinality of N is

$$\sharp N = \sum_{\substack{u \in Q_0 \\ u \text{ non-gentle}}} [X, M(1_u)][M(1_u), X]$$

for any $X \in \mathcal{F}(S)$. Let $(i, j, k, l, c) \in N$. There are arrows $\alpha, \beta, \gamma, \delta$ such that the sets $\mathrm{part}(\alpha^{-1}c\beta, b_k)$ and $\mathrm{part}(\gamma c\delta^{-1}, b_l)$ are non-empty. Applying Lemma 3.2 in case $k \neq l$ and Lemma 3.1 in case $k = l$, we obtain that at least one

of the words $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ cannot be a string. This can only happen if c is trivial. Let u be the vertex of Q with $c = 1_u$:

$$\begin{tikzcd} {} \arrow[dr, "\alpha"] & & {} \arrow[dl, "\beta"'] \\ & u \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ {} & & {} \end{tikzcd}$$

We apply the same lemmas once again: As the sets $\mathrm{part}(\beta^{-1}\alpha, b_k)$ and $\mathrm{part}(\gamma\delta^{-1}, b_l)$ are non-empty, we obtain that at least one of the words $\beta^{-1}\delta^{-1}$ and $\gamma\alpha$ cannot be a string. Therefore none of the pairs of words

- $(\delta\alpha, \gamma\beta)$
- $(\gamma\alpha, \delta\beta)$

can be a pair of strings. But this is only possible if the vertex u is non-gentle. For the cardinality of N we thus obtain

$$\begin{aligned} \sharp N &= \sum_{\substack{u\in Q_0 \\ u \text{ non-gentle}}} \sharp\,\mathrm{fac}(1_u, S)\sharp\,\mathrm{sub}(1_u, S) \\ &= \sum_{\substack{u\in Q_0 \\ u \text{ non-gentle}}} [S, 1_u][1_u, S] \\ &= \sum_{\substack{u\in Q_0 \\ u \text{ non-gentle}}} [X, M(1_u)][M(1_u), X] \end{aligned}$$

for any $X \in \mathcal{F}(S)$, which completes the proof. $\square$

4 Regular components of indecomposable modules

An $\mathcal{A}$-module $Y \in \operatorname{mod}(\mathcal{A}, d)$ is called a degeneration of $X \in \operatorname{mod}(\mathcal{A}, d)$ if Y belongs to the closure of the $\mathrm{GL}_d(k)$-orbit of X in $\operatorname{mod}(\mathcal{A}, d)$. In that case we also say that X degenerates to Y and write $X \leq_{\deg} Y$. We extend this notion to sequences of strings and quasi-bands: Let S and S' be finite sequences of strings and quasi-bands. We call S and S' equivalent, denoted by $S =_{\deg} S'$, if $\overline{\mathcal{F}(S)} = \overline{\mathcal{F}(S')}$, and we say that S degenerates to S', in symbols $S \leq_{\deg} S'$, if $\mathcal{F}(S') \subseteq \overline{\mathcal{F}(S)}$. Note that $\leq_{\deg}$ defines a partial order on the set of equivalence classes of finite sequences of strings and quasi-bands. Two sequences of strings and bands

$$S = (c_1, \dots, c_l, b_1, \dots, b_n) \text{ and } S' = (c'_1, \dots, c'_{l'}, b'_1, \dots, b'_{n'})$$

are equivalent if and only if $l = l'$, $n = n'$ and there are permutations $\sigma \in S_l$ and $\tau \in S_n$ satisfying

- $c'_{\sigma(i)} \in \{c_i, c_i^{-1}\}$ for $i = 1, \dots, l$ and
- $b'_{\tau(j)} \sim b_j$ for $j = 1, \dots, n$.

Note that this characterization might also hold for sequences of strings and quasi-bands, but we do not need it.

We establish two types of degenerations between sequences of bands, which yield a proof for Proposition 1.3.

Proof of Proposition 1.3. Let (b, m) be a band such that the closure of

$$\mathcal{F}(b, m) \subseteq \operatorname{mod}(\mathcal{A}, d)$$

is an irreducible component. If we assume that (b, m) is negligible, we can apply one of the following degenerations and obtain that $\mathcal{F}(b, m)$ is contained in the closure of another $\mathcal{S}$-family of quasi-band modules, which is impossible as $\overline{\mathcal{F}(b, m)}$ is an irreducible component. $\square$

The first degeneration can be described as follows: Cut off a piece of a suitable quasi-band, reverse the piece and reconnect it:

$$(b, m) = \begin{tikzcd} {} \arrow[d, "u"'] & {} \arrow[l, "w^{-1}"'] \\ {} \arrow[r, "w"'] & {} \arrow[u, "v"'] \end{tikzcd} \quad \rightsquigarrow \quad (c, m) = \begin{tikzcd} {} \arrow[d, "u"'] & {} \arrow[l, "w^{-1}"'] \arrow[d, "v"] \\ {} \arrow[r, "w"'] & {} \end{tikzcd}$$

Proposition 4.1. *Let (b, m) be a quasi-band and assume that there is a string u that starts and ends with an arrow, a string v that starts and ends with an inverse arrow and a string w such that $(b, m) = wuw^{-1}v$ and*

$$(c, m) := wuw^{-1}v^{-1}$$

is a quasi-band. Then $(c, m) <_{deg} (b, m)$.

Proof. Let $\tilde{Q}$ be the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\alpha_4"] \arrow[r, bend right, "\alpha_3"'] & 3 \arrow[l, bend right, "\alpha_2"'] \end{tikzcd}$$

and $\mathcal{V}$ be the variety of representations $X = (X(\alpha_1), X(\alpha_2), X(\alpha_3), X(\alpha_4))$ of $\tilde{Q}$ with dimension vector $(1, 2, 1)$, i.e.

$$\mathcal{V} = \mathrm{Mat}(2 \times 1, k) \times \mathrm{Mat}(2 \times 1, k) \times \mathrm{Mat}(1 \times 2, k) \times \mathrm{Mat}(1 \times 2, k).$$

For $\lambda, \mu \in k, \lambda \neq 0$, consider the representations

$$X_{\lambda,\mu} = (\begin{pmatrix} \lambda^{-1} \\ \mu \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} -\lambda\mu & 1 \end{pmatrix}) \in \mathcal{V},$$

$$Y_\nu := (\begin{pmatrix} 0 \\ \nu^{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}) \in \mathcal{V}.$$

The algebraic group $G = k^* \times \mathrm{GL}_2(k) \times k^*$ acts on $\mathcal{V}$ in the usual way, i.e.

$$(\varphi, \chi, \psi) \star (X_1, X_2, X_3, X_4) := (\chi X_1 \varphi^{-1}, \chi X_2 \psi^{-1}, \psi X_3 \chi^{-1}, \varphi X_4 \chi^{-1}).$$

For $\lambda, \mu \in k^*$ we apply the base change

$$g = (-\lambda^{-1}\mu^{-1}, \begin{pmatrix} 1 & -\lambda^{-1}\mu^{-1} \\ 0 & 1 \end{pmatrix}, 1)$$

to $X_{\lambda,\mu}$ and obtain $g \star X_{\lambda,\mu} = Y_{-\lambda^{-1}\mu^{-2}}$. Thus $X_{\lambda,\mu}$ belongs to a G-orbit of Y_ν for some $\nu \in k^*$, as long as $\mu \neq 0$.

Let A be the set of all paths of Q of length at most one, i.e. $A = Q_1 \cup \{1_x : x \in Q_0\}$. We identify the affine variety $\mathrm{mod}(\mathcal{A}, m)$ with a subvariety of $M_m(k)^A$. To show that $M(b, m, \lambda)$ belongs to the closure of $\mathcal{F}(c, m)$, we define a morphism

$$\phi : \mathcal{V} \longrightarrow M_m(k)^A$$

satisfying $\phi(X_{\lambda,\mu}) \in \mathcal{F}(c, m)$ for $\mu \neq 0$ and $\phi(X_{\lambda,0}) \simeq M(b, m, \lambda)$. Note that we will define ϕ in such a way that $\phi(\mathcal{V}) \subseteq \mathrm{mod}(kQ, m)$.

Here is the definition of ϕ: Let

$$X = (X(\alpha_1), X(\alpha_2), X(\alpha_3), X(\alpha_4)) \in \mathcal{V}$$

and let Z be the $\mathcal{A}$-module $Z = V \oplus M(w)^2 \oplus U$, where

$$V = \begin{cases} 0 & \text{if } l(v) = 1 \\ M(v') & \text{if } v = \beta^{-1}v'\alpha^{-1} \end{cases}$$

and

$$U = \begin{cases} 0 & \text{if } l(u) = 1 \\ M(u') & \text{if } u = \gamma u'\delta \end{cases}$$

We decompose U, V and $M(w)^2$ as k-vector spaces:

$$M(w)^2 = \bigoplus_{d \in \mathrm{Ld}(w)} W_d,$$

where $W_d = \mathrm{span}_k\{(e_d, 0), (0, e_d)\}$,

$$V = M(v') = V_{1_{t(v')}} \oplus \cdots \oplus V_{v'}$$

in case $v = \beta^{-1}v'\alpha^{-1}$, where $V_d := \mathrm{span}_k\{e_d\}$ for $d \in \mathrm{Ld}(v')$ and

$$U = U_{1_{t(u')}} \oplus \cdots \oplus U_{u'}$$

in case $u = \gamma u'\delta$, where $U_d = \mathrm{span}_k\{e_d\}$ for $d \in \mathrm{Ld}(u')$.

We identify $M_m(k)$ with $\mathrm{End}_k(Z)$, each W_d with k^2 and each U_d and V_d with k and set $\phi(X) = Z + Z_v + Z_u$, where the definition of $Z_v, Z_u \in \mathrm{End}_k(Z)$ depends on the lengths of u and v.

Case $v = \beta^{-1}v'\alpha^{-1}$:

$$\begin{tikzcd}
V_{v'} \arrow[dr, "Z_v(\alpha)=X(\alpha_1)"] \arrow[dd, dotted, no head] & & & \\
& W_{1_w} \arrow[r, dotted, no head] \arrow[dl, "Z_v(\beta)=X(\alpha_4)"] & W_w & U \\
V_{1_{t(v')}} & & &
\end{tikzcd}$$

Case $v = \alpha^{-1}$:

$$\begin{tikzcd}
W_{1_w} \arrow[loop above, "Z_v(\alpha)=X(\alpha_1)\circ X(\alpha_4)"] \arrow[r, dotted, no head] & W_w & U
\end{tikzcd}$$

Case $u = \gamma u'\delta$:

$$\begin{tikzcd}
 & & & U_{1_{t(u')}} \arrow[dl, "Z_u(\gamma)=X(\alpha_2)"'] \arrow[dd, dotted, no head] \\
V & W_{1_w} \arrow[r, dotted, no head] & W_w \arrow[dr, "Z_u(\delta)=X(\alpha_3)"'] & \\
 & & & U_{u'}
\end{tikzcd}$$

Case $u = \gamma$:

$$\begin{tikzcd}
V & W_{1_w} \arrow[r, dotted, no head] & W_w \arrow[loop above, "Z_u(\gamma)=X(\alpha_2)\circ X(\alpha_3)"]
\end{tikzcd}$$

By the definition of ϕ we have:

- $\phi(X_{\lambda,0}) \simeq M(b, m, \lambda) \in \mathcal{F}(b, m)$ for any $\lambda \in k^*$.
- $\phi(Y_\nu) \in \mathcal{F}(c, m)$ for any $\nu \in k^*$.

The morphism ϕ is G-equivariant with respect to the morphism of algebraic groups $G \longrightarrow \mathrm{GL}_m(k)$ sending (φ, χ, ψ) to

$$\begin{pmatrix} \varphi \cdot 1_V & & & & \\ & \chi & & & \\ & & \ddots & & \\ & & & \chi & \\ & & & & \psi \cdot 1_U \end{pmatrix}.$$

Therefore $\phi(X_\lambda, \mu)$ belongs to $\mathcal{F}(c, m)$ for $\mu \neq 0$ and thus $M(b, m, \lambda) \simeq \phi(X_{\lambda,0})$ belongs to the closure of $\mathcal{F}(c, m)$ for any $\lambda \in k^*$.

To complete the proof we show that

$$\overline{\mathcal{F}(b, m)} \neq \overline{\mathcal{F}(c, m)}.$$

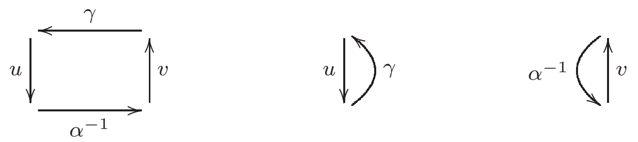
As $\sharp\,\mathrm{sub}(w, (b, m)) \neq \sharp\,\mathrm{sub}(w, (c, m))$, there is a string a with $[a, (b, m)] \neq [a, (c, n)]$ and thus minimum of the function

$$[M(a), -] : \mathrm{mod}(\mathcal{A}, d) \longrightarrow \mathbb{N}$$

on $\overline{\mathcal{F}(b, m)}$ differs from the minimum on $\overline{\mathcal{F}(c, m)}$, which implies that these two sets cannot be equal. $\square$

The second degeneration can be described as follows: Cut a suitable quasi-band into two pieces, and close each piece to separate quasi-bands.

$$(b,m) \qquad \rightsquigarrow \qquad (c,n) \qquad\qquad (d,m-n)$$



Proposition 4.2. *Let (b,m) be a quasi-band and assume that there are strings u,v,w,x,y, arrows $\alpha,\beta,\gamma,\delta$ and an $s \geq 1$ with*

$$(b,m) = u\gamma v\alpha^{-1} \text{ and } (b,sm) = w\beta x\alpha^{-1} = u\gamma w\delta^{-1}y$$

such that

$$(c,n) := u\gamma \text{ and } (d,m-n) := v\alpha^{-1}$$

are quasi-bands. Then $((c,n),(d,m-n)) <_{\deg} (b,m)$.

Proof. We only show that $((c,n),(d,m-n)) \leq_{\deg} (b,m)$ and leave the proof of the inequality $((c,n),(d,m-n)) \neq_{\deg} (b,m)$ to the reader, as it is nearly the same as in the proof of Proposition 4.1.

Let $\mu,\nu \in k^*$ and set $M := M(d,m-n,\mu)$ and $N := M(c,n,\nu)$. For any $h \in \mathrm{Hom}_k(M,N)$ there is a unique $\mathcal{A}$-module structure $X_{h,\mu,\nu}$ on the vector space $N \oplus M$ such that

$$\begin{pmatrix} 1_N & h \\ 0 & 1_M \end{pmatrix} : X_{h,\mu,\nu} \longrightarrow N \oplus M$$

is an $\mathcal{A}$-isomorphism. By definition, we know that

$$X_{h,\mu,\nu}(a) = \begin{pmatrix} N(a) & \zeta(a) \\ 0 & M(a) \end{pmatrix}$$

with $\zeta(a) = N(a)h - hM(a)$ for any $a \in \mathcal{A}$. Note that $\zeta(a) = 0$ for all $a \in \mathcal{A}$ in case h is $\mathcal{A}$-linear.

We will construct an $h \in \mathrm{Hom}_k(M,N)$ in such away that

$$\zeta(\epsilon)(\overline{1 \otimes e_i}) = \begin{cases} -\nu^{-1}\mu^{-1}\overline{1 \otimes e_0} & \text{if } i = m-n-1 \text{ and } \epsilon = \alpha, \\ \overline{1 \otimes e_{n-1}} & \text{if } i = 0 \text{ and } \epsilon = \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

for $0 \leq i < m-n$ and for any path ϵ of length at most one. Here is a picture of the module $X_{\mu,\nu} := X_{h,\mu,\nu}$:

$$
\begin{tikzcd}
\{0\} \times M(d, m-n, \mu) & (0, \overline{1 \otimes e_0}) \arrow[r, dash, "v"] \arrow[dr, "\gamma \simeq 1"' near start] & (0, \overline{1 \otimes e_{m-n-1}}) \arrow[l, bend right, "\alpha \simeq \mu^{-1}"'] \arrow[dl, "\alpha \simeq -\nu^{-1}\mu^{-1}" near start] \\
M(c, n, \nu) \times \{0\} & (\overline{1 \otimes e_0}, 0) \arrow[r, dash, "u"'] \arrow[r, bend right, "\gamma \simeq \nu"'] & (\overline{1 \otimes e_{n-1}}, 0)
\end{tikzcd}
$$

(An arrow from x to y labeled by $\alpha \simeq \lambda$ indicates that $X_{h,\mu,\nu}(\alpha)(x) = \lambda y$.)

We postpone the construction of h and show how to complete the proof, once h is defined: For a fixed $\lambda \in k^*$ the module $M(b, m, \lambda)$ belongs to the closure of the one-parameter family $Y_\mu := X_{\mu, -\lambda\mu^{-1}}$, $\mu \in k^*$. Consequently,

$$((c, n), (d, m-n)) \leq_{deg} (b, m).$$

In order to define h, we have to show that

$$c(1) \cdots c(l(w) + 1) = w\beta = b(1) \cdots b(l(w) + 1).$$

Let $0 < i \leq l(w) + 1$. Obviously $c(i) = b(i)$ if $i \leq n$. If $n < i$, we may assume inductively that $c(i-n) = b(i-n)$ and thus

$$c(i) = c(i-n) = b(i-n) = b(i).$$

Similarly, one can show that

$$d(1) \cdots d(l(w) + 1) = w\delta^{-1}.$$

There are integers $t, r \geq 0$ and strings z, z' such that

$$(c, tn) = w\beta z\gamma \text{ and } (d, r(m-n)) = w\delta^{-1}z'\alpha^{-1}.$$

Let $g : M(w) \longrightarrow M(c, tn, \nu^t)$ be the k-linear map sending e_d to $\overline{1 \otimes e_{l(d)}}$ for $d \in \mathrm{Ld}(w)$. We have

$$(g \circ \epsilon - \epsilon \circ g)(e_d) = \begin{cases} -\nu^t \overline{1 \otimes e_{tn-1}} & \text{if } d = 1_{t(w)} \text{ and } \epsilon = \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

for any path ϵ of length at most one and any $d \in \mathrm{Ld}(w)$. Similarly, let $f : M(d, r(m-n), \mu^r) \longrightarrow M(w)$ be the k-linear map sending $\overline{1 \otimes e_i}$ to

$$\begin{cases} e_d & \text{if } 0 \leq i \leq l(w), d \in \mathrm{Ld}(w), l(d) = i \\ 0 & \text{if } l(w) < i < r(m-n) \end{cases}$$

for $i = 0, \ldots, r(m-n) - 1$. The map f satisfies

$$(f \circ \epsilon - \epsilon \circ f)(\overline{1 \otimes e_i}) = \begin{cases} \mu^{-r} e_{1_{t(w)}} & \text{if } i = tn - 1 \text{ and } \epsilon = \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

for any path ϵ of length at most one and $0 \leq i < tn$. For the composition of f and g, we have

$$(g \circ f \circ \epsilon - \epsilon \circ g \circ f)(\overline{1 \otimes e_i}) = \begin{cases} \mu^{-r}\overline{1 \otimes e_0} & \text{if } i = tn - 1 \text{ and } \epsilon = \alpha, \\ -\nu^t \overline{1 \otimes e_{r(m-n)-1}} & \text{if } i = 0 \text{ and } \epsilon = \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

for any path ϵ of length at most one and $0 \leq i < r(m-n)$. Let h' be the composition

$$M(d, m-n, \mu) \longrightarrow M(d, r(m-n), \mu^r) \xrightarrow{g \circ f} M(c, tn, \nu^t) \longrightarrow M(c, n, \nu),$$

where the first map is an unwinding morphism and the last map is a winding morphism. We have

$$(h' \circ \epsilon - \epsilon \circ h')(\overline{1 \otimes e_i}) = \begin{cases} \mu^{-1}\overline{1 \otimes e_0} & \text{if } i = tn - 1 \text{ and } \epsilon = \alpha, \\ -\nu\overline{1 \otimes e_{m-n-1}} & \text{if } i = 0 \text{ and } \epsilon = \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

for any path ϵ of length at most one and any $0 \leq i < m-n$. Finally, we set $h := -\nu^{-1}h'$.

$\square$

5 Extensions

Let (b, m) and (c, n) be bands. Proposition 1.2 will follow from the following two lemmas.

Lemma 5.1. *If the pair* $([(b, m)], [(c, n)])$ *is not extendable, then*

$$\mathrm{Ext}^1_{\mathcal{A}}(M(b, m, \lambda), M(c, n, \mu)) = 0$$

for any $\lambda, \mu \in k^*$ *with* $\lambda \neq \mu, \mu^{-1}$.

Lemma 5.2. *If the pair* $((b, m), (c, n))$ *is extendable, there is a non-split short exact sequence of* $\mathcal{A}$*-modules*

$$0 \longrightarrow X \longrightarrow M_{X,Y} \longrightarrow Y \longrightarrow 0$$

with $M_{X,Y} \in \mathcal{F}(d, n+m)$ *for any* $X \in \mathcal{F}(c, n)$*,* $Y \in \mathcal{F}(b, m)$*, where*

$$(d, n+m) = c(1) \cdots c(n) b(1) \cdots b(m).$$

Recall from [6] that an $\mathcal{A}$-module A degenerates to a direct sum of $\mathcal{A}$-modules $B \oplus C$ whenever there is a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0.$$

Combining this result with Lemma 5.2, we obtain

Corollary 5.3. *If the pair* $((b, m), (c, n))$ *is extendable, then*

$$\mathcal{F}((c, n), (b, m)) \subseteq \overline{\mathcal{F}(d, n+m)},$$

where

$$(d, n+m) = c(1) \cdots c(n) b(1) \cdots b(m).$$

Proof of Lemma 5.1. By [2] the Auslander-Reiten translate $\tau^{-1}M$ of a band module M is isomorphic to M. It is well known that

$$\mathrm{Ext}^1_{\mathcal{A}}(N, M) \simeq \underline{\mathrm{Hom}}_{\mathcal{A}}(\tau^{-1}M, N)$$

for any finite dimensional $\mathcal{A}$-modules M, N, where

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\tau^{-1}M, N) = \mathrm{Hom}_{\mathcal{A}}(\tau^{-1}M, N)/\mathcal{P}(\tau^{-1}M, N)$$

and $\mathcal{P}(\tau^{-1}M, N)$ is the subspace of $\mathrm{Hom}_{\mathcal{A}}(\tau^{-1}M, N)$ consisting of all homomorphisms that factor through a projective $\mathcal{A}$-module. Therefore it suffices to show that any morphism $M(c, n, \mu) \longrightarrow M(b, m, \lambda)$ factors through a

projective $\mathcal{A}$-module. The condition $\lambda \neq \mu, \mu^{-1}$ asserts that $M(c, n, \mu)$ and $M(b, m, \lambda)$ are non-isomorphic and this allows us to apply Proposition 2.3, which yields a basis for

$$\operatorname{Hom}_{\mathcal{A}}(M(c, n, \mu), M(b, m, \lambda)).$$

We show that any morphism of the form

$$\iota_{j,w,(b,m),\lambda} \circ \pi_{i,w,(c,n),\mu} : M(c, n, \mu) \longrightarrow M(b, m, \lambda)$$

with $j \in \operatorname{sub}(w, (b, m))$ and $i \in \operatorname{fac}(w, (c, n))$ factors through a projective $\mathcal{A}$-module. Fix a string w, $j \in \operatorname{sub}(w, (b, m))$ and $i \in \operatorname{fac}(w, (c, n))$. Up to equivalence of bands, we may assume that $i = n$ and $j = m$. Moreover, we can assume that $m \in \operatorname{sub}_1(w, (b, m))$ and $n \in \operatorname{sub}_1(w, (c, n))$ up to replacing λ by λ^{-1} and μ by μ^{-1} if necessary. By the definition of the sets sub_1 and fac_1 there are $s, t \geq 1$, strings u, v and arrows $\alpha, \beta, \gamma, \delta$ with

$$(b, sm) = w\beta u\alpha^{-1} \text{ and } (c, tn) = w\delta^{-1}v\gamma.$$

As the pair $((b, m), (c, n))$ is not extendable,

$$(d, n + m) = c(1) \cdots c(n)b(1) \cdots b(m)$$

cannot be a quasi-band. As $c(n) = \gamma$ and $b(m) = \alpha^{-1}$ this can only happen if one of the words

$$c(1) \ldots c(n)b(1) \cdots b(m - 1) \text{ and } b(1) \cdots b(m)c(1) \cdots c(n - 1)$$

is not a string. We just consider the case, where the word

$$x := c(1) \cdots c(n)b(1) \cdots b(m - 1)$$

is not a string, as the other case is treated similarly. Let $1 \leq i \leq n$ be minimal with the property that $c(i), c(i + 1), \ldots, c(n)$ are arrows and let $1 \leq j \leq m - 1$ be maximal, such that $b(1), \ldots, b(j)$ are arrows. As x is not a string, we see that the path

$$c(i)c(i + 1) \cdots c(n)b(1) \cdots b(j)$$

belongs to the ideal I. We set

$$q = c(i)c(i + 1) \cdots c(n) \text{ and } r = b(1) \cdots b(j).$$

Obviously $w\beta \in \operatorname{Ld}(r)$, because otherwise $r \in \operatorname{Ld}(w)$, which implies that qr is a string. We may thus decompose $r = wr'$ for some non-trivial path

r'. Let $P = P_{s(r)}$ be the projective $\mathcal{A}$-module corresponding to the vertex $s(r)$. Obviously P is isomorphic to $M(q'rp^{-1})$ for some paths p and q' with $q = q''q'$ for a non-trivial path q''. We obtain the sequence of morphisms

$$M(c,n,\mu) \xrightarrow{h} M(q'w) \xrightarrow{g} P = M(q'wr'p^{-1}) \xrightarrow{f} M(b,m,\lambda),$$

where h is the factorstring morphism corresponding to

$$n - l(q') \in \mathrm{fac}_1(q'w,(c,n)),$$

g is the substring morphism corresponding to the decomposition

$$q'rp^{-1} = (1_{t(q')})\,(q'w)\,(r'p^{-1})$$

and f is the morphism sending $e_{q'r}$ to $\overline{1 \otimes e_{l(r)}}$. As

$$f \circ g \circ h = \iota_{j,w,(b,m),\lambda} \circ \pi_{i,w,(c,n),\mu},$$

the proof is complete.

$\square$

Proof of Lemma 5.2. As $((b,m),(c,n))$ is extendable, there are $s,t \geq 1$, strings u,v,w and arrows $\alpha,\beta,\gamma,\delta$ with

$$(b,sm) = w\beta u\alpha^{-1} \text{ and } (c,tn) = w\delta^{-1}v\gamma$$

such that

$$(d,n+m) := (c,n)(b,m) := c(1)\cdots c(n)b(1)\cdots b(m)$$

is a quasi-band.

Let $x = c(1)\cdots c(n)$ and $y = b(1)\cdots b(m)$ as strings. We divide the proof into four steps:

a) $l(w) < n+m$,

b) $n+m \in \mathrm{sub}(xw,(d,n+m))$,

c) $n \in \mathrm{fac}(yw,(d,n+m))$ and

d) for any $\lambda,\mu \in k^*$ the sequence

$$0 \longrightarrow M(c,n,\mu) \xrightarrow{f} M(d,n+m-\lambda\mu) \xrightarrow{g} M(b,m,\lambda) \longrightarrow 0,$$

where

$$f = \iota_{n+m,xw,(d,n+m),-\lambda\mu} \circ \pi_{n,xw,(c,n),\mu}$$

and

$$g = \iota_{m,yw,(b,m),\lambda} \circ \pi_{n,yw,(d,n+m),-\lambda\mu}$$

is exact and does not split.

Proof of a): If we assume that $l(w) \geq n+m$, we obtain the contradiction

$$\alpha^{-1} = b(m) = c(m) = c(n+m) = b(n+m) = b(n) = c(n) = \gamma.$$

Proof of b): As $d(n+m) = b(m) = \alpha^{-1}$ is an inverse arrow and $d(i) = c(i)$ for $i = 1, \ldots, n$, it suffices to show that $d(i+n) = b(i)$ for $i = 1, \ldots, l(w)+1$, as $b(l(w)+1) = \beta$ is an arrow. This is obvious if $i \leq m$. We may thus assume that $i > m$. As $l(w) < n+m$ by a), we see that $0 < i-m \leq n$ and $i - m \leq l(w)$ and thus

$$d(i+n) = d(i-m) = c(i-m) = b(i-m) = b(i).$$

Statement c) follows dually.

Proof of d): We decompose f as the sum $\sum\limits_{i=0}^{l(w)+n} f_i$ of k-linear maps

$$f_i : M(c, n, \mu) \longrightarrow M(d, n+m, -\lambda\mu).$$

Note that, in order to keep the coefficients combinatorially simple, we adapt the basis of $M(c, n, \mu)$ to i for the definition of f_i: For $0 \leq i \leq l(w) + n$ and $i \leq l < i + n$ we set

$$f_i(\overline{1 \otimes e_l}) := \begin{cases} \overline{1 \otimes e_i} & \text{if } i = l \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we decompose g as the sum $\sum\limits_{k=n}^{l(w)+m+n} g_k$ of k-linear maps

$$g_k : M(d, n+m, -\lambda\mu) \longrightarrow M(b, m, \lambda),$$

where g_k is defined as follows: For $n \leq k \leq l(w)+n+m$ and $k \leq l < k+n+m$ we set

$$g_k(\overline{1 \otimes e_l}) := \begin{cases} \overline{1 \otimes e_{l-n}} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 2.4, we see that f is injective and g surjective. In order to prove that the sequence is exact, it remains to show that $g \circ f = 0$. We have

$$g \circ f = \sum_{i=0}^{l(w)+n} \sum_{k=n}^{l(w)+m+n} g_k \circ f_i.$$

We claim that $g_k \circ f_i = 0$ unless (i, k) belongs to one of the disjoint sets

$$A := \{(i, i+n+m) : 0 \leq i \leq l(w)\}$$

and

$$B := \{(i,i) : n \le i \le l(w)+n\}.$$

Assume that there are $0 \le i \le l(w)+n$, $n \le k \le n+m+l(w)$ and $0 \le l < n$ such that $g_k \circ f_i(\overline{1 \otimes e_l}) \ne 0$. From the definition of f_i we obtain $l - i \in n\mathbb{Z}$ and $f_i(\overline{1 \otimes e_l}) = \xi\overline{(1 \otimes e_i)}$ for a $\xi \in k^*$ and from the definition of g_k we see that $k - i \in (n+m)\mathbb{Z}$. As $-l(w) \le k - i \le n + m + l(w)$ and $l(w) < n+m$, we obtain either $k-i = 0$ and thus $(i,k) \in B$ or $k-i = n+m$ and hence $(i,k) \in A$. We have

$$g \circ f = \sum_{(i,k)\in A} g_k \circ f_i + \sum_{(i,k)\in B} g_k \circ f_i = \sum_{(i,k)\in A} (g_k \circ f_i + g_{k-m} \circ f_{i+n})$$

and suffices to show that $(g_k \circ f_i + g_{k-m} \circ f_{i+n}) = 0$ for $(i,k) \in A$. Let $(i,k) \in A$. For any $i \le l < n+i$ we have

$$f_i(\overline{1 \otimes e_l}) = f_{i+n}(\overline{1 \otimes e_l}) = 0$$

and thus

$$g_k \circ f_i(\overline{1 \otimes e_l}) + g_{k-m} \circ f_{i+n}(\overline{1 \otimes e_l}) = 0,$$

unless $l = i$. In case $l = i$ we obtain

$$\begin{aligned} g_k f_i(\overline{1 \otimes e_i}) &= g_k(\overline{1 \otimes e_i}) \\ &= g_k(-\lambda\mu \cdot \overline{1 \otimes e_{i+n+m}}) \\ &= g_k(-\lambda\mu \cdot \overline{1 \otimes e_k}) \\ &= -\lambda\mu \cdot \overline{1 \otimes e_{k-n}} \end{aligned}$$

and

$$\begin{aligned} g_{k-m} f_{i+n}(\overline{1 \otimes e_i}) &= g_{k-m} f_{i+n}(\mu \cdot \overline{1 \otimes e_{i+n}}) \\ &= g_{k-m}(\mu \cdot \overline{1 \otimes e_{i+n}}) \\ &= g_{k-m}(\mu \cdot \overline{1 \otimes e_{k-m}}) \\ &= \mu \cdot \overline{1 \otimes e_{k-m-n}} \\ &= \lambda\mu \cdot \overline{1 \otimes e_{k-n}} \end{aligned}$$

and thus $g \circ f = 0$.

To complete the proof, it remains to show that the short exact sequence

$$0 \longrightarrow M(c,n,\mu) \xrightarrow{f} M(d, n+m, -\lambda\mu) \xrightarrow{g} M(b,m,\lambda) \longrightarrow 0,$$

does not split. It suffices to show that $M(d, n+m, -\lambda\mu)$ is not isomorphic to $M(c, n, \mu) \oplus M(b, m, \lambda)$. As

$$\sharp\,\mathrm{sub}(w,(c,n)) + \sharp\,\mathrm{sub}(w,(b,m)) > \sharp\,\mathrm{sub}(w,(d,n+m)),$$

there is an $\mathcal{A}$-module U, such that

$$[U, M(d, n+m, -\lambda\mu)] \neq [U, M(c, n, \mu) \oplus M(b, m, \lambda)],$$

by Proposition 2.2 about the homomorphism spaces between string and band modules. This shows that $M(d, n+m, -\lambda\mu)$ and $M(c, n, \mu) \oplus M(b, m, \lambda)$ cannot be isomorphic. $\square$

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